2.5 Uncertainty of Combined Permittivity and Permeability Determination

In this section an uncertainty analysis is presented. The sources of error in the permeability and permittivity TR measurement include

- Errors in measuring the magnitude and phase of the scattering parameters,
- Gaps between the sample and sample holder,
- Sample holder dimensional variations,
- Uncertainty in sample length,
- Line losses and connector mismatch, and
- Uncertainty in reference plane positions.

A technique for correcting errors arising from gaps around the sample is given in Appendix B [33,34,35]. Gaps between holder and sample either may be corrected using the formulas given in the appendix or conducting liquid solder can be painted on the external surfaces of the sample that are in contact with the sample holder before insertion into the sample holder, thereby minimizing gap problems. The formulas given in the literature generally under-correct for the real part of the permittivity and over-correct for the imaginary part of the permittivity. We assume that all measurements of permittivity have been corrected for air gaps around the sample before the uncertainty analysis is applied. In order to evaluate the uncertainty introduced by the measured scattering parameters and sample dimensions, a differential uncertainty analysis is assumed applicable with the uncertainty due to S_{11} and S_{21} evaluated separately. We assume that the S-parameters are functions of $S_{ij}(|S_{11}|, |S_{21}|, \theta_{11}, \theta_{21}, L, d)$. We assume that the total uncertainty in ϵ'_R, where d is the air gap between the sample and waveguide. We assume that the uncertainties for the physically measured parameters are

$$\frac{\Delta\epsilon'_R}{\epsilon'_R} = \frac{1}{\epsilon'_R}\sqrt{\sum_\alpha \left[\left(\frac{\partial\epsilon'_R}{\partial|S_\alpha|}\Delta|S_\alpha|\right)^2 + \left(\frac{\partial\epsilon'_R}{\partial\theta_\alpha}\Delta\theta_\alpha\right)^2\right] + \left(\frac{\partial\epsilon'_R}{\partial L}\Delta L\right)^2 + \left(\frac{\partial\epsilon'_R}{\partial d}\Delta d\right)^2}\,, \qquad (2.67)$$

$$\frac{\Delta\epsilon''_R}{\epsilon''_R} = \frac{1}{\epsilon''_R}\sqrt{\sum_\alpha \left[\left(\frac{\partial\epsilon''_R}{\partial|S_\alpha|}\Delta|S_\alpha|\right)^2 + \left(\frac{\partial\epsilon''_R}{\partial\theta_\alpha}\Delta\theta_\alpha\right)^2\right] + \left(\frac{\partial\epsilon''_R}{\partial L}\Delta L\right)^2 + \left(\frac{\partial\epsilon''_R}{\partial d}\Delta d\right)^2}\,, \qquad (2.68)$$

where $\alpha = 11$ or 21, $\Delta\theta$ is the uncertainty in the phase of the scattering parameter, $\Delta|S_\alpha|$ is the uncertainty in the magnitude of the scattering parameter, Δd is the uncertainty in the air gap around the sample, and ΔL is the uncertainty in the sample length. The derivatives with respect to air gap, $\partial\epsilon'_R/\partial d$, have been presented previously [26]. The uncertainties used for the S-parameters depend on the specific ANA used for the measurements. This type of uncertainty analysis assumes that changes in independent variables are sufficiently small so that a Taylor series expansion is valid. Of course there are many other uncertainty sources of lesser magnitude such as repeatability of connections and torquing of flange bolts. Estimates for these uncertainties could be added to the uncertainty budget.

2.5.1 One Sample at One Position

For the uncertainty analysis it is necessary to take implicit derivatives of the S-parameter equations with respect to the assumed independent parameters. It is assumed that the functions are analytic over the region of interest with respect to the differentiation variables. The independent variables are assumed to be $|S_{21}|$, $|S_{11}|$, θ_{11}, θ_{21}, and L. The derivatives of the S-parameter eqs (2.31) through (2.33) can be found analytically

$$\frac{\partial S_{11}}{\partial Z}[\frac{\partial Z}{\partial \epsilon_R^*}\frac{\partial \epsilon_R^*}{\partial |S_{11}|} + \frac{\partial Z}{\partial \mu_R^*}\frac{\partial \mu_R^*}{\partial |S_{11}|}] + \frac{\partial S_{11}}{\partial \Gamma}[\frac{\partial \Gamma}{\partial \epsilon_R^*}\frac{\partial \epsilon_R^*}{\partial |S_{11}|} + \frac{\partial \Gamma}{\partial \mu_R^*}\frac{\partial \mu_R^*}{\partial |S_{11}|}] = \exp(j\theta_{11}) \ , \tag{2.69}$$

$$\frac{\partial S_{11}}{\partial Z}[\frac{\partial Z}{\partial \epsilon_R^*}\frac{\partial \epsilon_R^*}{\partial |S_{21}|} + \frac{\partial Z}{\partial \mu_R^*}\frac{\partial \mu_R^*}{\partial |S_{21}|}] + \frac{\partial S_{11}}{\partial \Gamma}[\frac{\partial \Gamma}{\partial \epsilon_R^*}\frac{\partial \epsilon_R^*}{\partial |S_{21}|} + \frac{\partial \Gamma}{\partial \mu_R^*}\frac{\partial \mu_R^*}{\partial |S_{21}|}] = 0 \ , \tag{2.70}$$

$$\frac{\partial S_{21}}{\partial Z}[\frac{\partial Z}{\partial \epsilon_R^*}\frac{\partial \epsilon_R^*}{\partial |S_{11}|} + \frac{\partial Z}{\partial \mu_R^*}\frac{\partial \mu_R^*}{\partial |S_{11}|}] + \frac{\partial S_{21}}{\partial \Gamma}[\frac{\partial \Gamma}{\partial \epsilon_R^*}\frac{\partial \epsilon_R^*}{\partial |S_{11}|} + \frac{\partial \Gamma}{\partial \mu_R^*}\frac{\partial \mu_R^*}{\partial |S_{11}|}] = 0 \ , \tag{2.71}$$

$$\frac{\partial S_{21}}{\partial Z}[\frac{\partial Z}{\partial \epsilon_R^*}\frac{\partial \epsilon_R^*}{\partial |S_{21}|} + \frac{\partial Z}{\partial \mu_R^*}\frac{\partial \mu_R^*}{\partial |S_{21}|}] + \frac{\partial S_{21}}{\partial \Gamma}[\frac{\partial \Gamma}{\partial \epsilon_R^*}\frac{\partial \epsilon_R^*}{\partial |S_{21}|} + \frac{\partial \Gamma}{\partial \mu_R^*}\frac{\partial \mu_R^*}{\partial |S_{21}|}] = \exp(j\theta_{21}) \ . \tag{2.72}$$

We can rewrite eq (2.69) - (2.72) as

$$\underbrace{(\frac{\partial S_{11}}{\partial Z}\frac{\partial Z}{\partial \epsilon_R^*} + \frac{\partial S_{11}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \epsilon_R^*})}_{A}\frac{\partial \epsilon_R^*}{\partial |S_{11}|} +$$
$$\underbrace{(\frac{\partial S_{11}}{\partial Z}\frac{\partial Z}{\partial \mu_R^*} + \frac{\partial S_{11}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \mu_R^*})}_{B}\frac{\partial \mu_R^*}{\partial |S_{11}|} = \exp(j\theta_{11}) \ , \tag{2.73}$$

$$\underbrace{\left(\frac{\partial S_{11}}{\partial Z}\frac{\partial Z}{\partial \epsilon_R^*} + \frac{\partial S_{11}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \epsilon_R^*}\right)}_{A}\frac{\partial \epsilon_R^*}{\partial |S_{21}|} + \underbrace{\left(\frac{\partial S_{11}}{\partial Z}\frac{\partial Z}{\partial \mu_R^*} + \frac{\partial S_{11}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \mu_R^*}\right)}_{B}\frac{\partial \mu_R^*}{\partial |S_{21}|} = 0 , \tag{2.74}$$

$$\underbrace{\left(\frac{\partial S_{21}}{\partial Z}\frac{\partial Z}{\partial \epsilon_R^*} + \frac{\partial S_{21}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \epsilon_R^*}\right)}_{C}\frac{\partial \epsilon_R^*}{\partial |S_{11}|} + \underbrace{\left(\frac{\partial S_{21}}{\partial Z}\frac{\partial Z}{\partial \mu_R^*} + \frac{\partial S_{21}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \mu_R^*}\right)}_{D}\frac{\partial \mu_R^*}{\partial |S_{11}|} = 0 , \tag{2.75}$$

$$\underbrace{\left(\frac{\partial S_{21}}{\partial Z}\frac{\partial Z}{\partial \epsilon_R^*} + \frac{\partial S_{21}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \epsilon_R^*}\right)}_{C}\frac{\partial \epsilon_R^*}{\partial |S_{21}|} + \underbrace{\left(\frac{\partial S_{21}}{\partial Z}\frac{\partial Z}{\partial \mu_R^*} + \frac{\partial S_{21}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \mu_R^*}\right)}_{D}\frac{\partial \mu_R^*}{\partial |S_{21}|} = \exp(j\theta_{21}) , \tag{2.76}$$

where we have defined parameters A, B, C, and D. If we let

$$E = -\frac{\partial S_{11}}{\partial Z}\frac{\partial Z}{\partial L} , \tag{2.77}$$

$$F = -\frac{\partial S_{21}}{\partial Z}\frac{\partial Z}{\partial L} , \tag{2.78}$$

we can solve for the derivatives that have been taken with respect to the independent parameters in eqs (2.73)- (2.75):

$$\frac{\partial \epsilon_R^*}{\partial |S_{11}|} = \frac{\exp(j\theta_{11})}{[A - \frac{BC}{D}]} , \tag{2.79}$$

$$\frac{\partial \mu_R^*}{\partial |S_{11}|} = -\frac{C}{D}\frac{\partial \epsilon_R^*}{\partial |S_{11}|} , \tag{2.80}$$

$$\frac{\partial \epsilon_R^*}{\partial L} = \frac{BF - DE}{BC - AD} , \tag{2.81}$$

$$\frac{\partial \mu_R^*}{\partial L} = \frac{E - A\frac{\partial \epsilon_R^*}{\partial L}}{B} , \tag{2.82}$$

$$\frac{\partial \epsilon_R^*}{\partial \theta_{11}} = j|S_{11}|\frac{\partial \epsilon_R^*}{\partial |S_{11}|} , \tag{2.83}$$

$$\frac{\partial \mu_R^*}{\partial \theta_{11}} = j|S_{11}|\frac{\partial \mu_R^*}{\partial |S_{11}|} , \tag{2.84}$$

$$\frac{\partial \epsilon_R^*}{\partial \theta_{21}} = j|S_{21}|\frac{\partial \epsilon_R^*}{\partial |S_{21}|} , \tag{2.85}$$

$$\frac{\partial \mu_R^*}{\partial \theta_{21}} = j|S_{21}|\frac{\partial \mu_R^*}{\partial |S_{21}|} , \tag{2.86}$$

$$\frac{\partial S_{11}}{\partial \Gamma} = -\frac{(1+\Gamma^2 Z^2)(Z^2-1)}{(1-\Gamma^2 Z^2)^2} , \tag{2.87}$$

$$\frac{\partial S_{11}}{\partial Z} = \frac{2Z\Gamma(\Gamma^2-1)}{(1-\Gamma^2 Z^2)^2} , \tag{2.88}$$

$$\frac{\partial S_{21}}{\partial Z} = \frac{(1-\Gamma^2)(Z^2\Gamma^2+1)}{(1-\Gamma^2 Z^2)^2} , \tag{2.89}$$

$$\frac{\partial S_{21}}{\partial \Gamma} = \frac{2Z\Gamma(Z^2-1)}{(1-\Gamma^2 Z^2)^2} , \tag{2.90}$$

$$\frac{\partial \Gamma}{\partial \epsilon_R^*} = \frac{\gamma_0 \mu_R^{2*} \epsilon_0 \mu_0 \omega^2}{\gamma(\gamma + \gamma_0 \mu_R^*)^2} , \tag{2.91}$$

$$\frac{\partial \Gamma}{\partial \mu_R^*} = \frac{\epsilon_R^*}{\mu_R^*}\frac{\partial \Gamma}{\partial \epsilon_R^*} + \frac{2\gamma_0 \gamma}{(\gamma + \gamma_0 \mu_R^*)^2} , \tag{2.92}$$

$$\frac{\partial Z}{\partial L} = -\gamma \exp(-\gamma L) , \tag{2.93}$$

$$\frac{\partial Z}{\partial \epsilon_R^*} = \frac{L\mu_R^* \omega^2 \epsilon_0 \mu_0}{2\gamma} \exp(-\gamma L) , \tag{2.94}$$

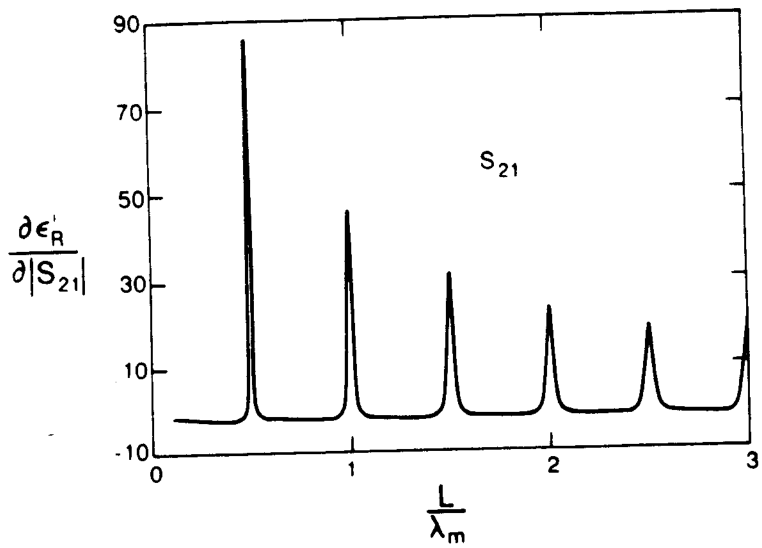

Figure 2.18: The derivative of ϵ_R' by $|S_{21}|$ vs L/λ_m with $\epsilon_R^* = (5.0, 0.02)$, $\mu_R^* = (2.0, 0.03)$.

$$\frac{\partial Z}{\partial \mu_R^*} = \frac{L\epsilon_R^* \omega^2 \epsilon_0 \mu_0}{2\gamma} \exp(-\gamma L) \ . \tag{2.95}$$

The measurement bounds for S-parameter data are obtained from specifications for a network analyzer. The dominant uncertainty is in the phase of S_{11} as $| S_{11} | \to 0$. The uncertainty in $|S_{21}|$ is relatively constant until $|S_{21}| \leq$ -50 dB, when it increases abruptly. The various derivatives are plotted in figures 2.18 through 2.27.

In figures 2.28 through 2.31 the total uncertainty in ϵ_R^* and μ_R^* computed from S_{21} and S_{11} is plotted as a function of normalized sample length. For low-loss and high-loss materials at 3 GHz with various values of ϵ_R^* and the guided wavelength in the material given by

$$\lambda_m = \frac{2\pi}{\sqrt{\omega^2 \frac{(\sqrt{\epsilon'^2 + \epsilon''^2} + \epsilon')}{2} \mu' - (\frac{2\pi}{\lambda_c})^2}} \ . \tag{2.96}$$

In figures 2.28 through 2.31 the error due to the gap correction is not included, nor are there uncertainties included for connector repeatability or flange bolt torquing. The maximum uncertainty for low-loss materials occurs at multiples of one-half wavelength. Generally, we see a decrease in uncertainty as a function of increasing sample length. Also, the uncertainties in the S-parameters have some frequency dependence with higher frequencies having larger uncertainties in phase.

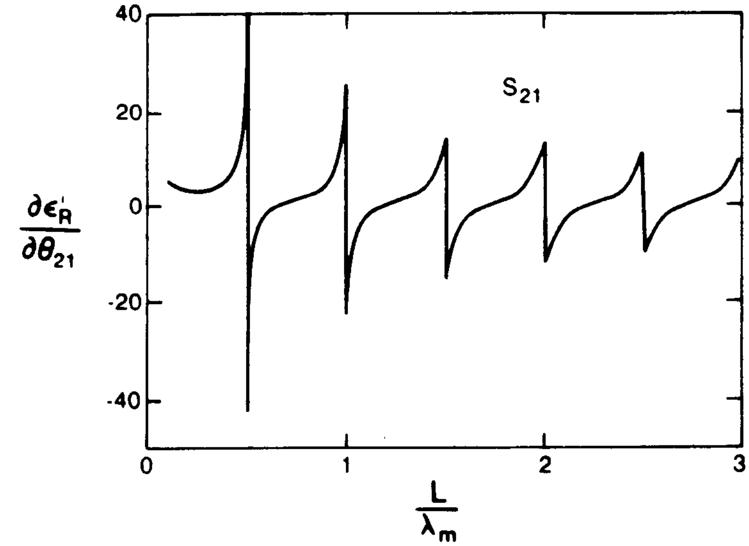

Figure 2.19: The derivative of ϵ'_R by θ_{21} using S_{21} vs L/λ_m with $\epsilon^*_R = (5.0, 0.01)$, $\mu^*_R = (2.0, 0.03)$.

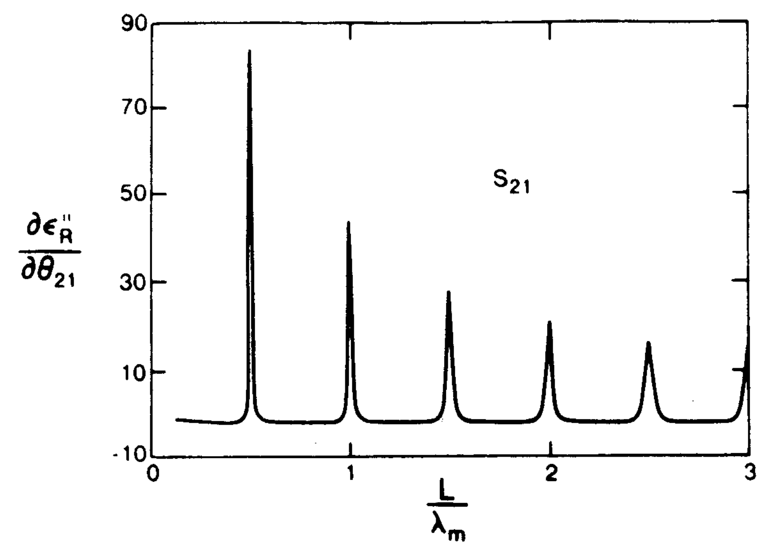

Figure 2.20: The derivative of ϵ''_R with respect to θ_{21} using S_{21} with $\epsilon^*_R = (5.0, 0.01), \mu^*_R = (2.0, 0.03)$.

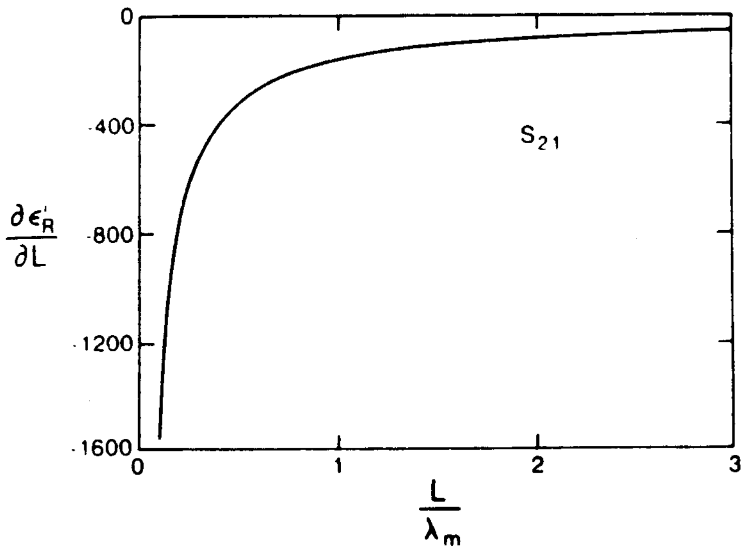

Figure 2.21: The derivative of ϵ_R' with respect to L using S_{21} with $\epsilon_R^* = (5.0, 0.01)$, $\mu_R^* = (2.0, 0.03)$.

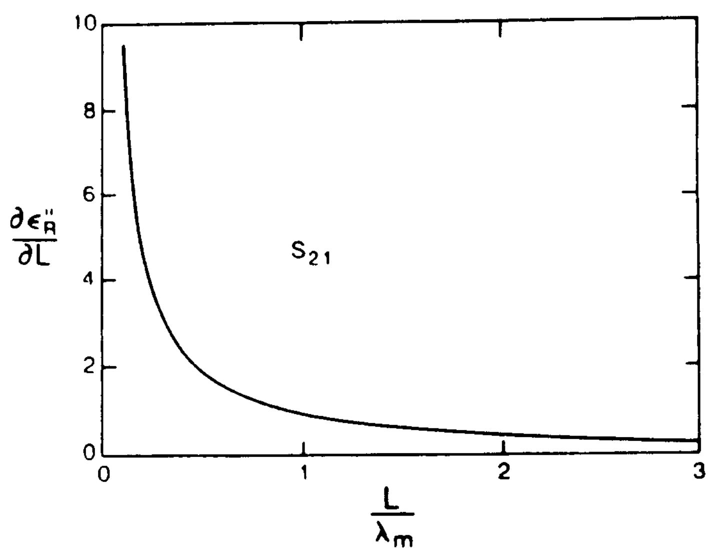

Figure 2.22: The derivative of ϵ_R'' with respect to L using S_{21} with $\epsilon_R^* = (5.0, 0.01), \mu_R^* = (2.0, 0.03)$.

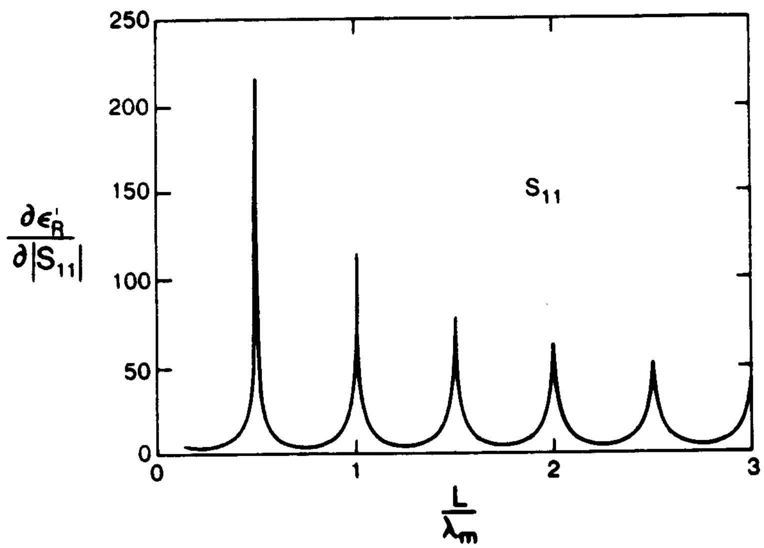

Figure 2.23: The derivative of ϵ_R' with respect to $|S_{11}|$ with $\epsilon_R^* = (5.0, 0.01), \mu_R^* = (2.0, 0.03)$.

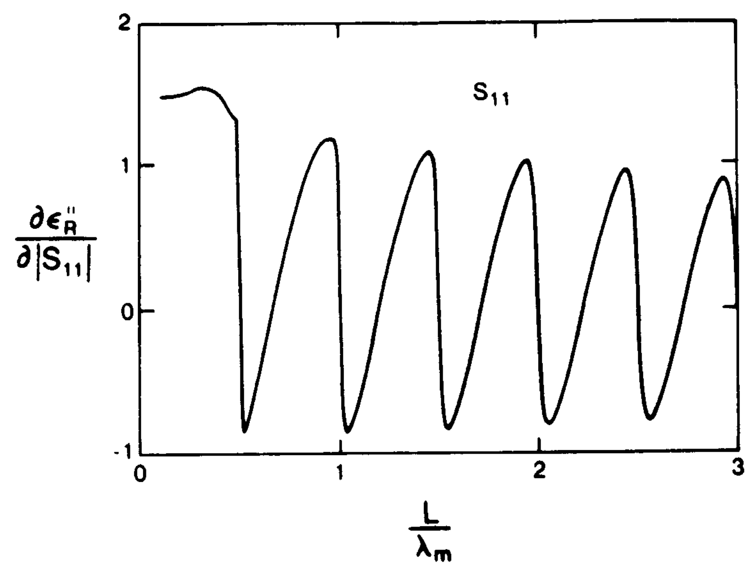

Figure 2.24: The derivative of ϵ_R'' with respect to $|S_{11}|$ with $\epsilon_R^* = (5.0, 0.01), \mu_R^* = (2.0, 0.03)$.

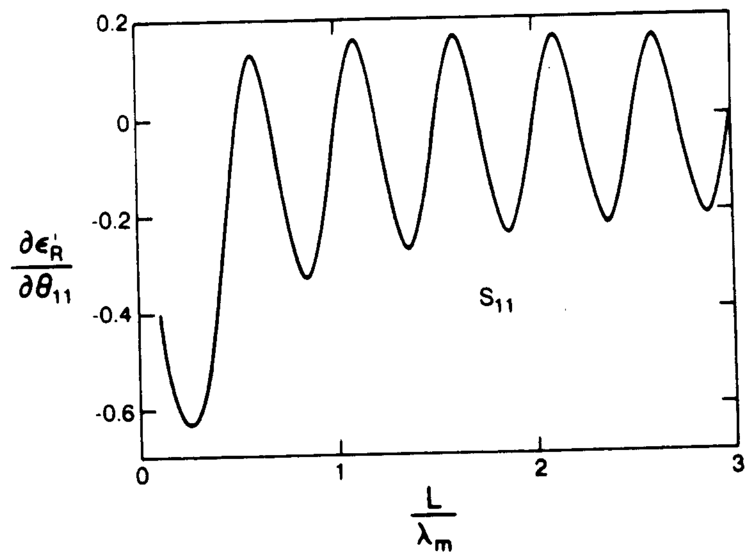

Figure 2.25: The derivative of ϵ'_R with respect to θ_{11} using S_{11} with $\epsilon^*_R = (5.0, 0.01), \mu^*_R = (2.0, 0.03)$.

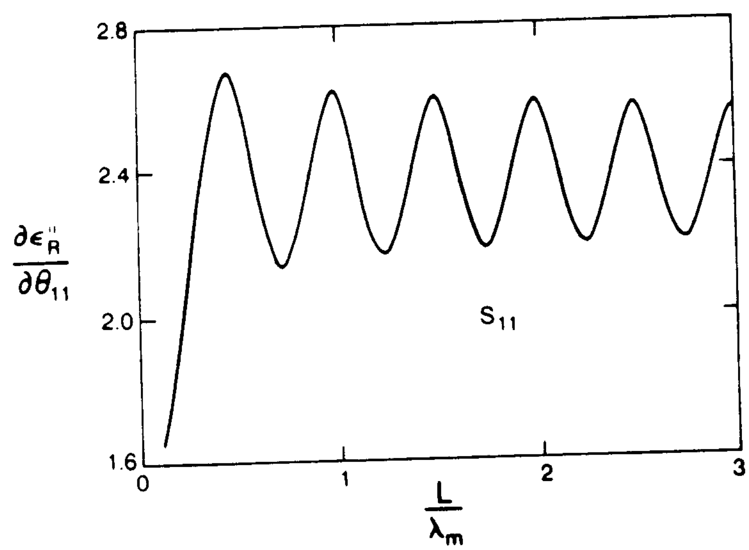

Figure 2.26: The derivative of ϵ''_R with respect to θ_{11} using S_{11} with $\epsilon^*_R = (5.0, 0.01), \mu^*_R = (2.0, 0.03)$.

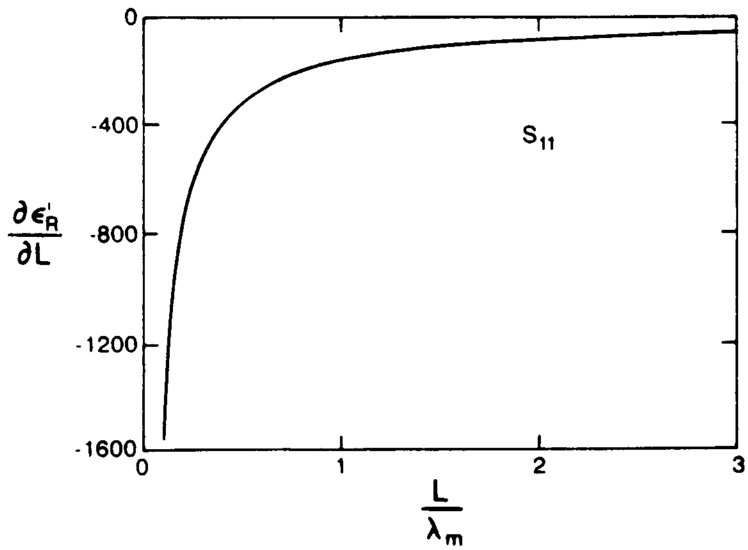

Figure 2.27: The derivative of ϵ_R' with respect to L using S_{11} with $\epsilon_R^* = (5.0, 0.01), \mu_R^* = (2.0, 0.03)$.

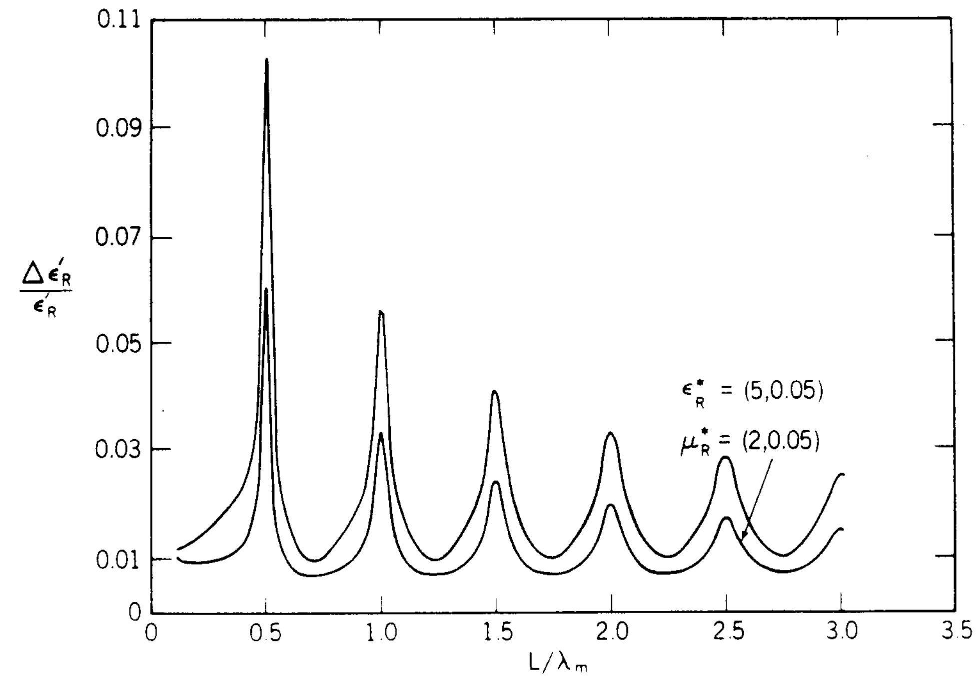

Figure 2.28: The relative uncertainty in $\epsilon_R'(\omega)$ for a low-loss material as a function of normalized length, with $\mu_R^* = (2, 0.05)$, $\epsilon_R^* = (10, 0.05)$ and $(5, 0.05)$.

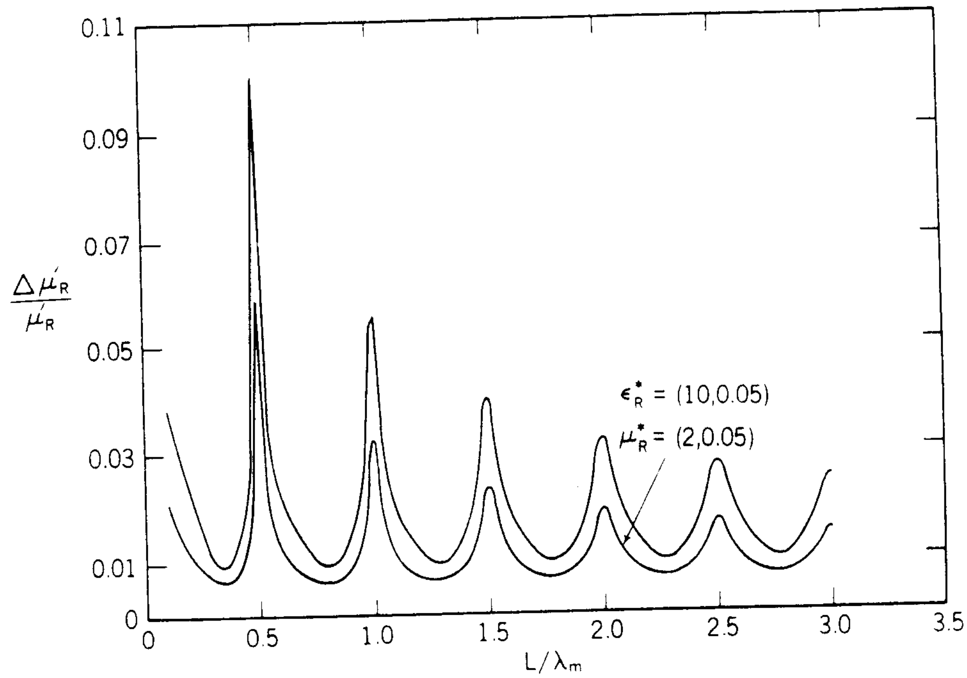

Figure 2.29: The relative uncertainty in $\mu'_R(\omega)$ for a low-loss material as a function of normalized length, with $\mu^*_R = (2, 0.05)$, $\epsilon^*_R = (10, 0.05)$ and $(5, 0.05)$.

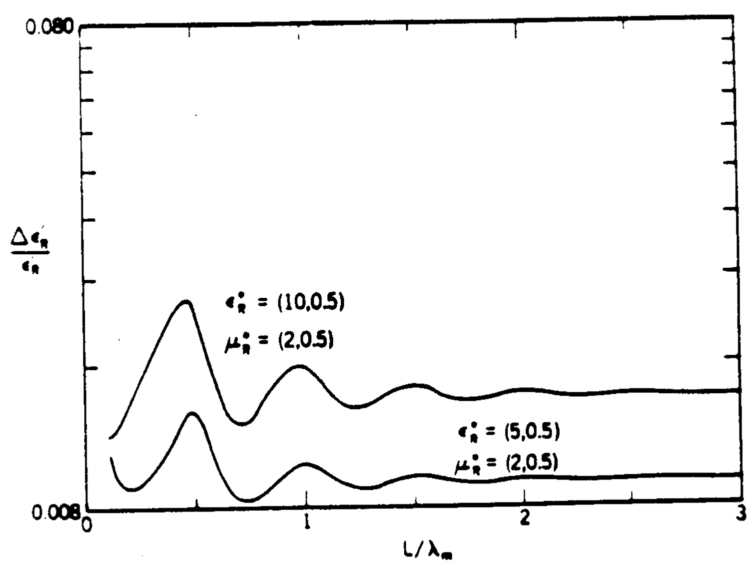

Figure 2.30: The relative uncertainty in $\epsilon'_R(\omega)$ for a high-loss material as a function of normalized length, with $\mu^*_R = (2, 0.5)$, $\epsilon^*_R = (10, 0.5)$ and $(5, 0.5)$.

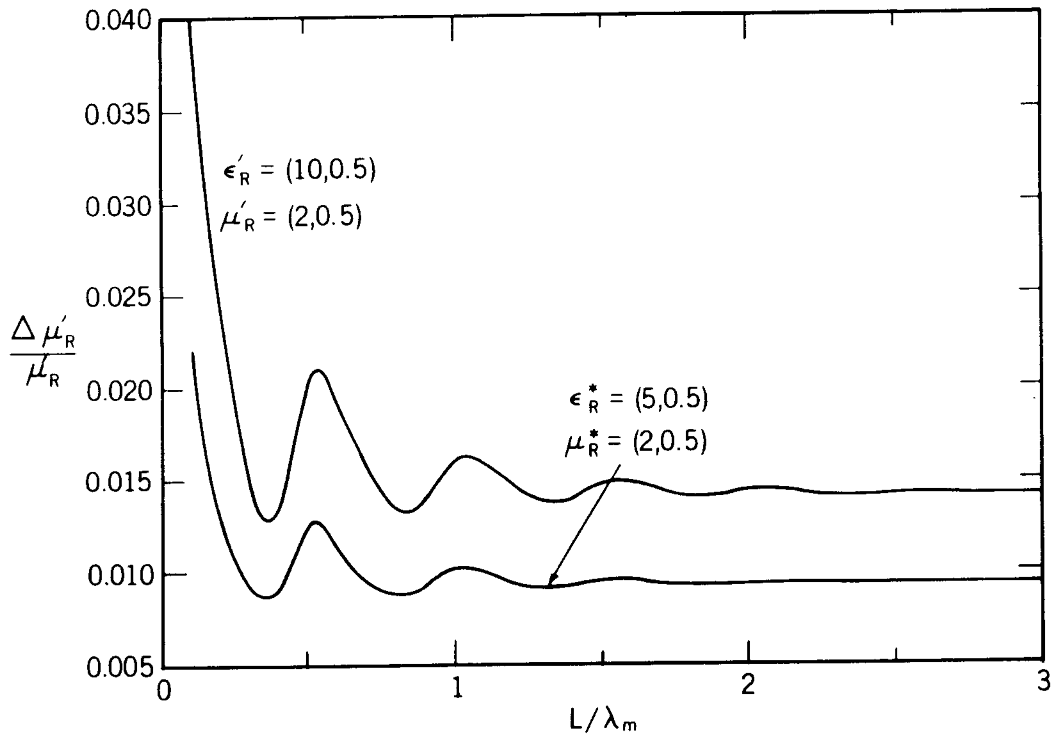

Figure 2.31: The relative uncertainty in $\mu'_R(\omega)$ for a high-loss material as a function of normalized length, with $\mu^*_R = (2.0, 0.5)$, $\epsilon^*_R = (10.0, 0.5)$ and $(5.0, 0.5)$.

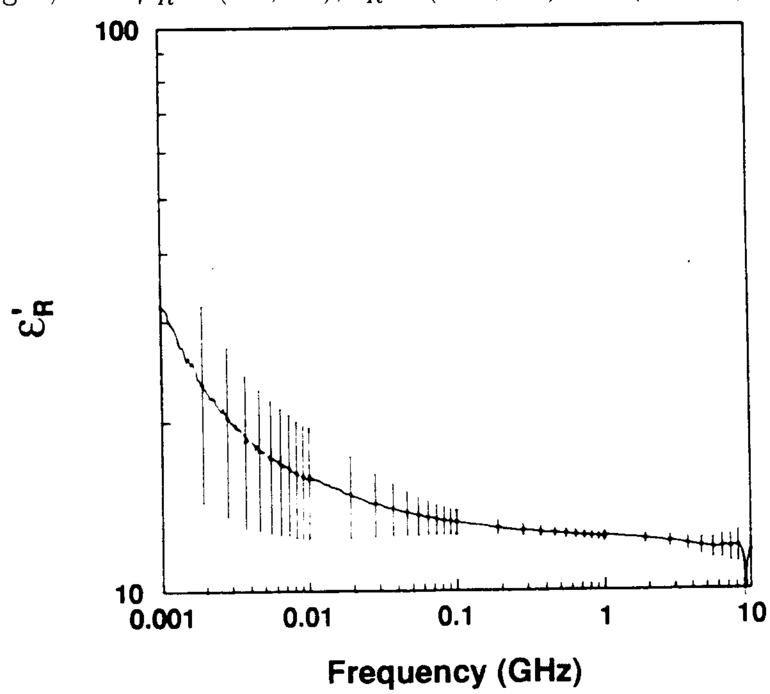

Figure 2.32: The real part of the relative permittivity $\epsilon'_R(\omega)$ for a nickel-zinc compound with uncertainties.

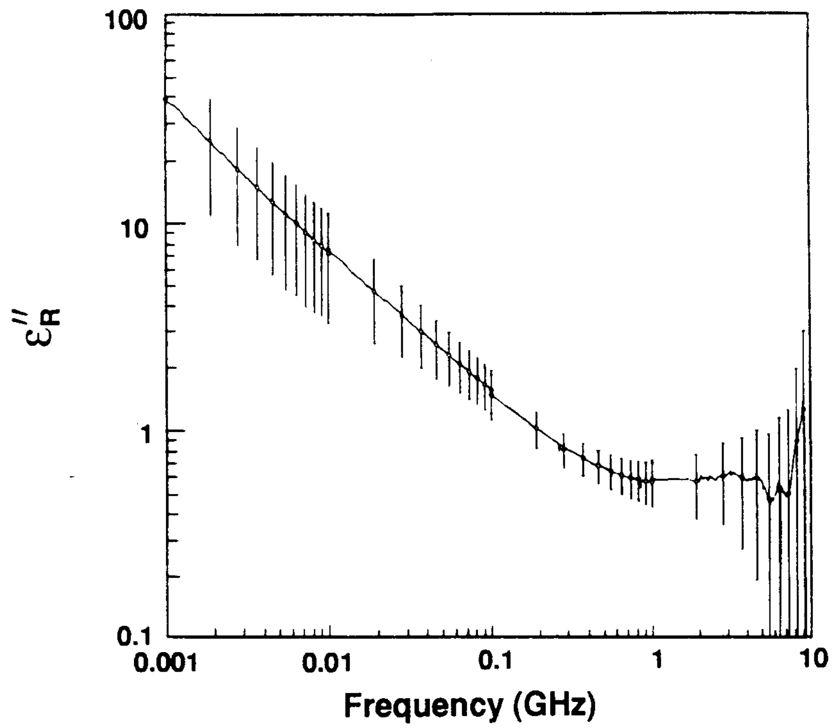

Figure 2.33: The imaginary part of the relative permittivity $\epsilon''_R(\omega)$ for a nickel-zinc compound with uncertainties.

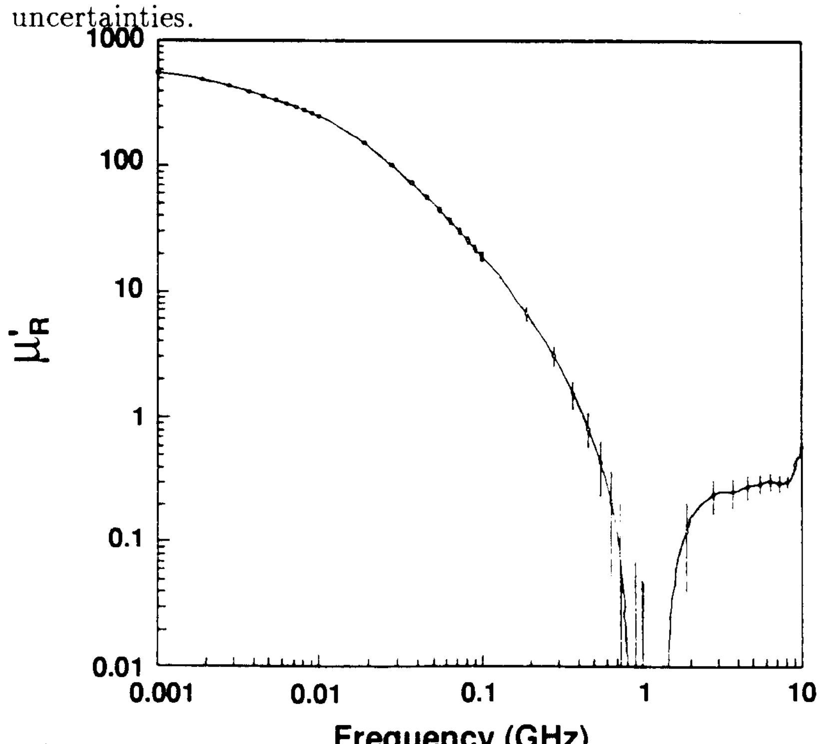

Figure 2.34: The real part of the relative permeability $\mu'_R(\omega)$ for a nickel-zinc compound with uncertainties.

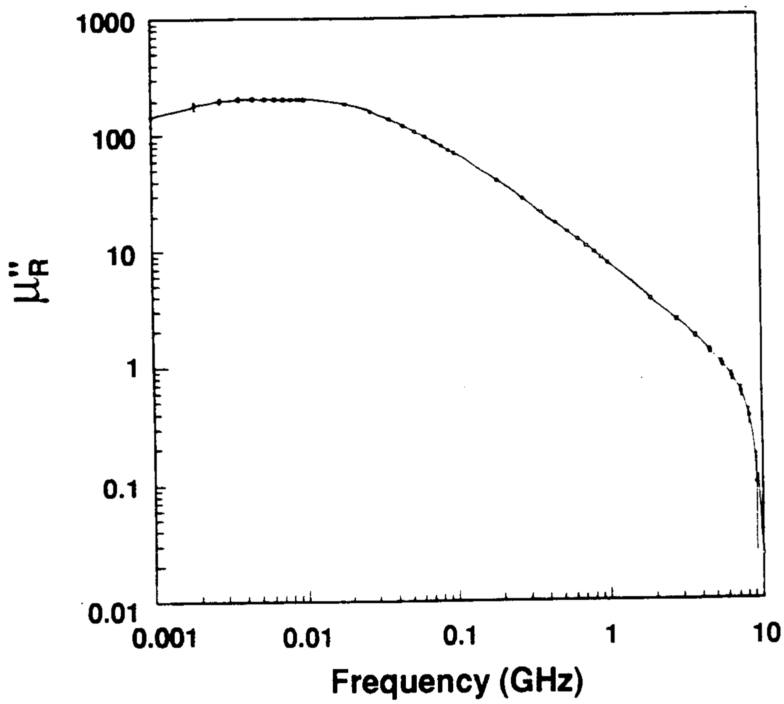

Figure 2.35: The imaginary part of the relative permeability $\mu''_R(\omega)$ for a nickel-zinc compound with uncertainties.

In figures 2.32 through 2.35 a measurement of a nickel-zinc ferrite compound is given with associated uncertainties. Uncertainties increase at high and low frequencies. At high and low frequency extremes the uncertainties in phase increase. Also, the scale is logarithmic which distorts the lengths of the error bars.

2.5.2 Two Samples of Differing Lengths

Another method to determine permittivity and permeability is the measurement of two samples with differing lengths. The advantage of this method is that each sample resonates at a different frequency and therefore S_{11} can be appreciable over the entire frequency band.

We assume the S-parameters are functions of $S_{ij}(|S_{mn}|, \theta_{mn}, L_1, L_2)$. The parameters used for measurements on materials of low to medium loss are

$$S_{21(i)} = \frac{Z_i(1-\Gamma^2)}{1-Z_i^2\Gamma^2} , \tag{2.97}$$

since it is acceptable down to -40 dB. We assume that the lengths of the samples are L_1 and $L_2 = \alpha L_1$. Due to the two lengths, there are transmission coefficients for each sample

$$Z_1 = \exp(-\gamma L_1) , \tag{2.98}$$

$$Z_2 = \exp(-\alpha\gamma L_1) . \tag{2.99}$$

The relevant partial derivatives of eqs (2.97) are:

$$\begin{aligned} &\frac{\partial S_{21(1)}}{\partial Z_1}[\frac{\partial Z_1}{\partial \epsilon_R^*}\frac{\partial \epsilon_R^*}{\partial |S_{21(1)}|} + \frac{\partial Z_1}{\partial \mu_R^*}\frac{\partial \mu_R^*}{\partial |S_{21(1)}|}] + \\ &\frac{\partial S_{21(1)}}{\partial \Gamma}[\frac{\partial \Gamma}{\partial \epsilon_R^*}\frac{\partial \epsilon_R^*}{\partial |S_{21(1)}|} + \frac{\partial \Gamma}{\partial \mu_R^*}\frac{\partial \mu_R^*}{\partial |S_{21(1)}|}] \\ &= \exp(j\theta_1) , \end{aligned} \tag{2.100}$$

$$\frac{\partial S_{21(2)}}{\partial Z_2}[\frac{\partial Z_2}{\partial \epsilon_R^*}\frac{\partial \epsilon_R^*}{\partial |S_{21(1)}|} + \frac{\partial Z_2}{\partial \mu_R^*}\frac{\partial \mu_R^*}{\partial |S_{21(1)}|}] + \frac{\partial S_{21(2)}}{\partial \Gamma}[\frac{\partial \Gamma}{\partial \epsilon_R^*}\frac{\partial \epsilon_R^*}{\partial |S_{21(1)}|} + \frac{\partial \Gamma}{\partial \mu_R^*}\frac{\partial \mu_R^*}{\partial |S_{21(1)}|}] = 0 , \tag{2.101}$$

$$\frac{\partial S_{21(1)}}{\partial Z_1}[\frac{\partial Z_1}{\partial \epsilon_R^*}\frac{\partial \epsilon_R^*}{\partial |S_{21(2)}|} + \frac{\partial Z_1}{\partial \mu_R^*}\frac{\partial \mu_R^*}{\partial |S_{21(2)}|}] + \frac{\partial S_{21(1)}}{\partial \Gamma}[\frac{\partial \Gamma}{\partial \epsilon_R^*}\frac{\partial \epsilon_R^*}{\partial |S_{21(2)}|} + \frac{\partial \Gamma}{\partial \mu_R^*}\frac{\partial \mu_R^*}{\partial |S_{21(2)}|}] = 0 , \tag{2.102}$$

$$\frac{\partial S_{21(2)}}{\partial Z_2}[\frac{\partial Z_2}{\partial \epsilon_R^*}\frac{\partial \epsilon_R^*}{\partial |S_{21(2)}|}+\frac{\partial Z_2}{\partial \mu_R^*}\frac{\partial \mu_R^*}{\partial |S_{21(2)}|}]$$
$$+\frac{\partial S_{21(2)}}{\partial \Gamma}[\frac{\partial \Gamma}{\partial \epsilon_R^*}\frac{\partial \epsilon_R^*}{\partial |S_{21(2)}|}+\frac{\partial \Gamma}{\partial \mu_R^*}\frac{\partial \mu_R^*}{\partial |S_{21(2)}|}]$$
$$= \exp(j\theta_2) \ . \tag{2.103}$$

We can rewrite eqs (2.100) through (2.103) as

$$\underbrace{(\frac{\partial S_{21(1)}}{\partial Z_1}\frac{\partial Z_1}{\partial \epsilon_R^*}+\frac{\partial S_{21(1)}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \epsilon_R^*})}_{A_1}\frac{\partial \epsilon_R^*}{\partial |S_{21(1)}|}$$
$$+\underbrace{(\frac{\partial S_{21(1)}}{\partial Z_1}\frac{\partial Z}{\partial \mu_R^*}+\frac{\partial S_{21(1)}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \mu_R^*})}_{B_1}\frac{\partial \mu_R^*}{\partial |S_{21(1)}|} = \exp(j\theta_1) \ , \tag{2.104}$$

$$\underbrace{(\frac{\partial S_{21(2)}}{\partial Z_2}\frac{\partial Z_2}{\partial \epsilon_R^*}+\frac{\partial S_{21(2)}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \epsilon_R^*})}_{A_2}\frac{\partial \epsilon_R^*}{\partial |S_{21(1)}|}$$
$$+\underbrace{(\frac{\partial S_{21(2)}}{\partial Z_2}\frac{\partial Z_2}{\partial \mu_R^*}+\frac{\partial S_{21(2)}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \mu_R^*})}_{B_2}\frac{\partial \mu_R^*}{\partial |S_{21(1)}|} = 0 \ , \tag{2.105}$$

$$\underbrace{(\frac{\partial S_{21(1)}}{\partial Z_1}\frac{\partial Z_1}{\partial \epsilon_R^*}+\frac{\partial S_{21(1)}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \epsilon_R^*})}_{A_1}\frac{\partial \epsilon_R^*}{\partial |S_{21(2)}|}$$
$$+\underbrace{(\frac{\partial S_{21(1)}}{\partial Z_1}\frac{\partial Z_1}{\partial \mu_R^*}+\frac{\partial S_{21(1)}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \mu_R^*})}_{B_1}\frac{\partial \mu_R^*}{\partial |S_{21(2)}|} = 0 \ , \tag{2.106}$$

$$\underbrace{(\frac{\partial S_{21(2)}}{\partial Z_2}\frac{\partial Z_2}{\partial \epsilon_R^*} + \frac{\partial S_{21(2)}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \epsilon_R^*})}_{A_2}\frac{\partial \epsilon_R^*}{\partial |S_{21(2)}|}$$

$$+\underbrace{(\frac{\partial S_{21(2)}}{\partial Z_2}\frac{\partial Z_2}{\partial \mu_R^*} + \frac{\partial S_{21(2)}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \mu_R^*})}_{B_2}\frac{\partial \mu_R^*}{\partial |S_{21(2)}|} = \exp(j\theta_2)\ , \qquad (2.107)$$

where we have defined parameters A_1, B_1, A_2, and B_2. Also for the relevant derivatives with respect to length, we find

$$\underbrace{(\frac{\partial S_{21(1)}}{\partial Z_1}\frac{\partial Z_1}{\partial \epsilon_R^*} + \frac{\partial S_{21(1)}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \epsilon_R^*})}_{A_1}\frac{\partial \epsilon_R^*}{\partial L_1}$$

$$+\underbrace{(\frac{\partial S_{21(1)}}{\partial Z_1}\frac{\partial Z_1}{\partial \mu_R^*} + \frac{\partial S_{21(1)}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \mu_R^*})}_{B_1}\frac{\partial \mu_R^*}{\partial L_1} + \underbrace{\frac{\partial S_{21(1)}}{\partial Z_1}\frac{\partial Z_1}{\partial L_1}}_{E_1} = 0\ , \qquad (2.108)$$

$$\underbrace{(\frac{\partial S_{21(2)}}{\partial Z_2}\frac{\partial Z_2}{\partial \epsilon_R^*} + \frac{\partial S_{21(2)}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \epsilon_R^*})}_{A_2}\frac{\partial \epsilon_R^*}{\partial L_1}$$

$$+\underbrace{(\frac{\partial S_{21(2)}}{\partial Z_2}\frac{\partial Z_2}{\partial \mu_R^*} + \frac{\partial S_{21(2)}}{\partial \Gamma}\frac{\partial \Gamma}{\partial \mu_R^*})}_{B_2}\frac{\partial \mu_R^*}{\partial L_1} + \underbrace{\frac{\partial S_{21(2)}}{\partial Z_2}\frac{\partial Z_2}{\partial L_1}}_{F2} = 0\ . \qquad (2.109)$$

We now can solve for the derivatives that have been taken with respect to the independent parameters in eqs (2.104) through (2.107)

$$\frac{\partial \epsilon_R^*}{\partial |S_{21(1)}|} = \frac{B_2 \exp(j\theta_1)}{A_1 B_2 - B_1 A_2}\ , \qquad (2.110)$$

$$\frac{\partial \mu_R^*}{\partial |S_{21(1)}|} = -\frac{A_2}{B_2}\frac{\partial \epsilon_R^*}{|S_{21(1)}|}\ , \qquad (2.111)$$

$$\frac{\partial \epsilon_R^*}{\partial |S_{21(2)}|} = \frac{B_1 \exp(j\theta_2)}{A_2 B_1 - B_2 A_1}\ , \qquad (2.112)$$

$$\frac{\partial \mu_R^*}{\partial |S_{21(2)}|} = -\frac{A_1}{B_1}\frac{\partial \epsilon_R^*}{|S_{21(2)}|}\ , \qquad (2.113)$$

$$\frac{\partial \epsilon_R^*}{\partial L_1} = \frac{E_1 B_2 - F_2 B_1}{B_1 A_2 - A_1 B_2} , \tag{2.114}$$

$$\frac{\partial \mu_R^*}{\partial L_1} = -\frac{E_1 + A_1 \frac{\partial \epsilon_R^*}{\partial L}}{B_1} , \tag{2.115}$$

$$\frac{\partial \epsilon_R^*}{\partial \theta_1} = j|S_{21(1)}| \frac{\partial \epsilon_R^*}{\partial |S_{21(1)}|} , \tag{2.116}$$

$$\frac{\partial \mu_R^*}{\partial \theta_1} = j|S_{21(1)}| \frac{\partial \mu_R^*}{\partial |S_{21(1)}|} , \tag{2.117}$$

$$\frac{\partial \epsilon_R^*}{\partial \theta_2} = j|S_{21(2)}| \frac{\partial \epsilon_R^*}{\partial |S_{21(2)}|} , \tag{2.118}$$

$$\frac{\partial \mu_R^*}{\partial \theta_2} = j|S_{21(2)}| \frac{\partial \mu_R^*}{\partial |S_{21(2)}|} , \tag{2.119}$$

$$\frac{\partial S_{21(i)}}{\partial Z_i} = \frac{(1 + \Gamma^2 Z_i^2)(Z_i^2 - 1)}{(1 - \Gamma^2 Z_i^2)^2} , \tag{2.120}$$

$$\frac{\partial S_{21(i)}}{\partial \Gamma} = \frac{2 Z_i \Gamma (Z_i^2 - 1)}{(1 - \Gamma^2 Z_i^2)^2} , \tag{2.121}$$

$$\frac{\partial \Gamma}{\partial \epsilon_R^*} = \frac{\gamma_0 \mu_R^{2*} \epsilon_0 \mu_0 \omega^2}{\gamma (\mu_0 \gamma + \gamma_0 \mu)^2} , \tag{2.122}$$

$$\frac{\partial \Gamma}{\partial \mu_R^*} = \frac{\epsilon_R^*}{\mu_R^*} \frac{\partial \Gamma}{\partial \epsilon_R^*} + \frac{2 \gamma_0 \gamma}{(\gamma + \gamma_0 \mu)^2} , \tag{2.123}$$

$$\frac{\partial Z}{\partial L} = -\gamma \exp(-\gamma L) , \tag{2.124}$$

$$\frac{\partial Z}{\partial \epsilon_R^*} = \frac{L \epsilon_0 \mu \omega^2}{2\gamma} \exp(-\gamma L) , \tag{2.125}$$

$$\frac{\partial Z}{\partial \mu_R^*} = \frac{L \epsilon \mu_0 \omega^2}{2\gamma} \exp(-\gamma L) . \tag{2.126}$$

In figures 2.36 through 2.37, the total uncertainty in ϵ_R^* and μ_R^* computed from S_{21} is plotted as a function of normalized sample length, for low-loss and high-loss materials at 3 GHz with various values of ϵ_R^*.

When the length of one sample is twice the length of the other sample, we see instability at frequencies corresponding to $n\lambda_m/2$. Generally, we see a decrease in uncertainty as a

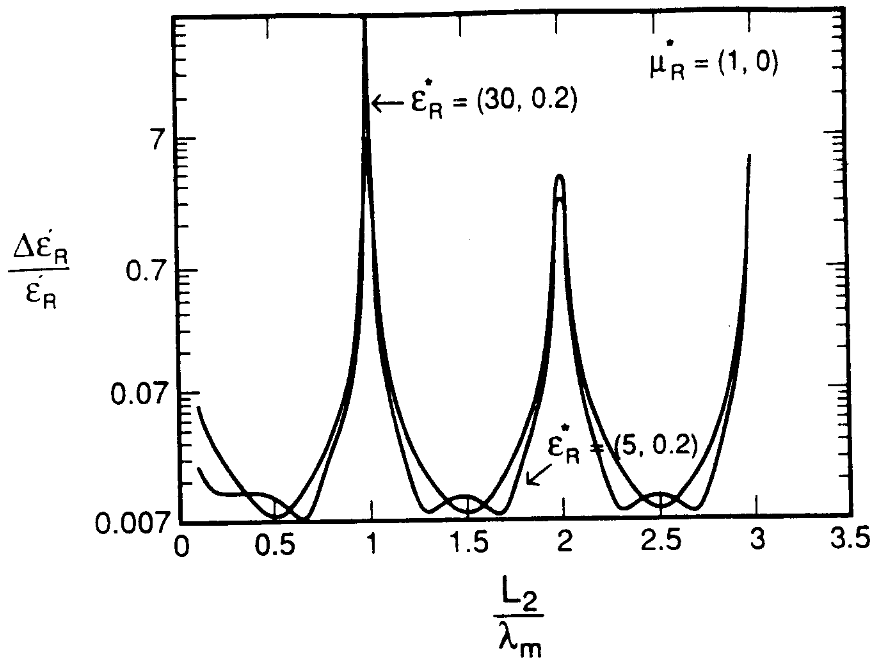

Figure 2.36: The relative uncertainty in $\epsilon_R'(\omega)$ for a low-loss material as a function of normalized length for the case when $L_1 = 0.5L_2$ for two different permittivities.

function of increasing sample length. Also, the uncertainties in the S-parameters show some frequency dependence. In figure 2.37 the ratio of sample lengths is $\sqrt{2}$. In this case we see greater stability over the frequency range than in the case where the range is 0.5. Resonances in the solutions will occur when $L = n\lambda_m/2$ and $\alpha L = m\lambda_m/2$ simultaneously, where m is an integer.

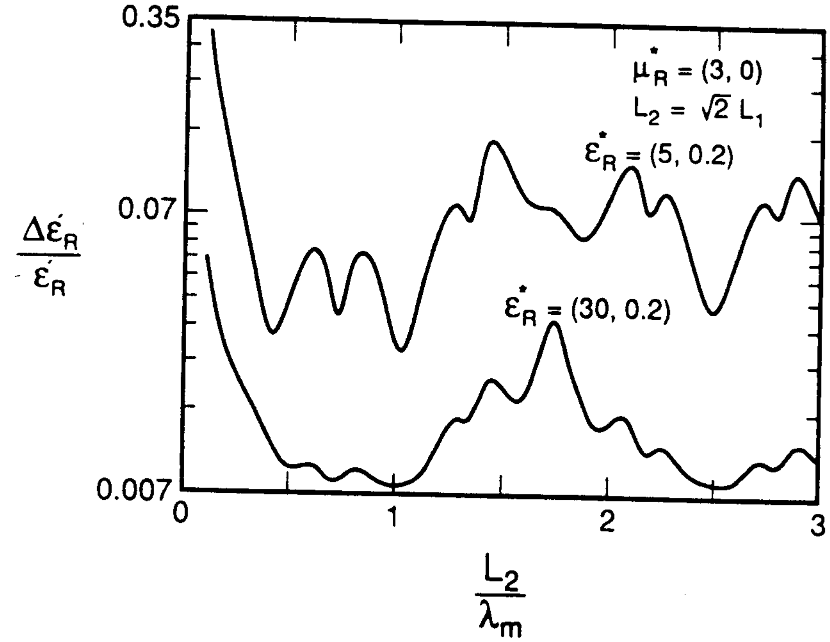

Figure 2.37: The relative uncertainty in $\epsilon'_R(\omega)$ for a low-loss material as a function of normalized length for the case when $L_2^2 = 2L_1^2$ for two different permittivities.

2.6 Uncertainty in Gap Correction

The correction for an air gap between the wall of the sample holder and sample is very important for measurements of high permittivity materials. In addition, the uncertainty in the gap correction is very important for high permittivity materials and may actually dominant the uncertainties of the measurement. In appendix C the gap correction is worked out in detail. In this section the uncertainty in the gap correction will be worked out.

2.6.1 Dielectric Materials

Waveguide Gap Uncertainty

The uncertainty due to an air gap between sample and holder can be calculated from the partial derivatives of ϵ_R^* with respect to sample thicknesses, d. The relevant derivatives for waveguide are given by

$$\frac{\partial \epsilon'_{cR}}{\partial d} = \epsilon'_{mR}[\frac{1}{b-(b-d)\epsilon'_{mR}}] - \epsilon'^2_{mR}\frac{d}{[b-(b-d)\epsilon'_{mR}]^2} , \tag{2.127}$$

$$\frac{\partial \epsilon''_{cR}}{\partial d} = -\epsilon''_{mR}\epsilon'_{R}\frac{b}{[b-(b-d)\epsilon'_{mR}]^2} . \tag{2.128}$$

Coaxial Gap Correction

For coaxial line the relevant derivatives are given by

$$\frac{\partial \epsilon'_{cR}}{\partial R_2} = -\epsilon'_{mR}\frac{1}{R_2(L_3-\epsilon'_{mR}L_1)} + \epsilon'^2_{mR}\frac{L_2}{R_2(L_3-\epsilon'_{mR}L_1)^2} , \tag{2.129}$$

$$\frac{\partial \epsilon'_{cR}}{\partial R_3} = \epsilon'_{mR}\frac{1}{R_3(L_3-\epsilon'_{mR}L_1)} + \epsilon'^2_{mR}\frac{L_2}{R_3(L_3-\epsilon'_{mR}L_1)^2} , \tag{2.130}$$

$$\frac{\partial \epsilon''_{cR}}{\partial R_2} = \epsilon''_{mR}\epsilon'_{mR}[\frac{1}{L_2R_2} + \frac{L_1}{L_2^2R_2}] , \tag{2.131}$$

$$\frac{\partial \epsilon''_{cR}}{\partial R_3} = -\epsilon''_{mR}\epsilon'_{mR}[\frac{1}{L_2R_3} + \frac{L_1}{L_2^2R_3}] . \tag{2.132}$$

2.6.2 Magnetic Materials

Waveguide Gap Uncertainty

The uncertainty due to an air gap between sample and holder can be calculated from the partial derivatives of μ_R^* with respect to gap thicknesses, d. The relevant derivatives for waveguide are given by

$$\frac{\partial \mu'_{cR}}{\partial d} = (1 - \mu'_{mR})\frac{b}{d^2} , \tag{2.133}$$

$$\frac{\partial \mu''_{cR}}{\partial d} = -\mu'_{mR}\frac{b}{d^2} . \tag{2.134}$$

Coaxial Gap Correction

For coaxial line the relevant derivatives are calculated using

$$K_1 = \ln R_2/R_1 , \tag{2.135}$$

$$K_2 = \ln R_3/R_2 , \tag{2.136}$$

$$K_3 = \ln R_4/R_3 , \tag{2.137}$$

$$K_4 = \ln R_4/R_1 , \tag{2.138}$$

as follows

$$\frac{\partial \mu'_{cR}}{\partial R_2} = \frac{1}{R_2 K_2}[\mu'_{mR}\frac{K_4}{K_2} - \frac{K_1}{K_2} - 1 - \frac{K_3}{K_2}] , \tag{2.139}$$

$$\frac{\partial \mu'_{cR}}{\partial R_3} = \frac{1}{R_3 K_2}[-\mu'_{mR}\frac{K_4}{K_2} + \frac{K_1}{K_2} + 1 + \frac{K_3}{K_2}] , \tag{2.140}$$

$$\frac{\partial \mu''_{cR}}{\partial R_2} = \mu''_{mR}\frac{K_1}{R_2 K_2^2} , \tag{2.141}$$

$$\frac{\partial \mu''_{cR}}{\partial R_3} = -\mu''_{mR}\frac{K_1}{R_3 K_2^2} . \tag{2.142}$$

2.6.3 Higher Order Modes

The field model assumes a single mode of propagation in the sample. Propagation of higher order modes becomes possible in inhomogeneous samples of high dielectric constant due to changes in cutoff. Air gaps also play an important role in mode conversion. Generally, the appearance of higher order modes manifests itself as a sudden dip in $|S_{11}|$. This dip is a result of resonance of the excited higher order mode. We can expect point-by-point

TR models to break down near higher order mode resonances for materials of high dielectric constant or inhomogeneous samples. Optimized, multi-frequency solution techniques fare better in this respect. The characteristic of the higher order modes are anomalies in the scattering matrix at and around resonance. Higher order modes require a coupling mechanism in order to begin propagating. In waveguide and coaxial line the asymmetry of the sample promotes higher order mode propagation. In order to minimize the effects of higher order modes, shorter samples can be used to maintain the electrical length less than one-half guided wavelength. Also well machined sample are important in suppressing modes. Higher order modes will not appear if the sample length is less than one-half guided wavelength of the fundamental mode in the material.

Mode Suppression

It is possible to remove some of the higher order modes by mode filters. This would be particularly helpful in cylindrical waveguide. One way to do this is to helically wind a fine wire about the inner surface of the waveguide sample holder, thus eliminating longitudinal currents and therefore TM modes. Another approach is to insert cuts in the waveguide walls to minimize current loops around the waveguide.